Eur. Phys. J. B **14**, 691–698 (2000)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Exact and approximate results of non-extensive quantum statistics

U. Tırnaklı[1,2,a] and D.F. Torres[3]

[1] Centro Brasileiro de Pesquisas Fisicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
[2] Department of Physics, Faculty of Science, Ege University 35100 Izmir-Turkey
[3] Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900, La Plata, Buenos Aires, Argentina

Received 19 August 1999 and Received in final form 1 November 1999

Abstract. We develop an analytical technique to derive explicit forms of thermodynamical quantities within the asymptotic approach to non-extensive quantum distribution functions. Using it, we find an expression for the number of particles in a boson system which we compare with other approximate scheme (*i.e.* factorization approach), and with the recently obtained exact result. To do this, we investigate the predictions on Bose-Einstein condensation and the blackbody radiation. We find that both approximation techniques give results similar to (up to $\mathcal{O}(q-1)$) the exact ones, making them a useful tool for computations. Because of the simplicity of the factorization approach formulae, it appears that this is the easiest way to handle with physical systems which might exhibit slight deviations from extensivity.

PACS. 05.20.-y Classical statistical mechanics – 05.30.Jp Boson systems – 05.30.Fk Fermion systems and electron gas

1 Introduction

Since the papers by Tsallis [1,2], non-extensive statistical formalism has been shown to be not only *robust* –it allows generalizations of all necessary fundamental concepts of thermostatistics [3]–, but also *useful* –it provides a suitable theoretical tool to explain some of the experimental situations where standard thermostatistics has shortcomings, due to the presence of long-range interactions, or long-range memory effects, or (multi)-fractal space-time constraints. See reference [4] for a periodically updated bibliography.

The core of this generalized formalism is defined through a generalized entropy

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}, \quad (q \in \Re), \tag{1}$$

where k is a positive constant, $\{p_i\}$ is a set of probabilities and W is the total number of microscopic configurations. It is easy to verify that the $q \to 1$ limit immediately recovers the usual (extensive) Boltzmann-Gibbs entropy. Moreover, if a composed system $A+B$ has probabilities which factorize into those corresponding to the subsystems A and B, then $S_q(A+B)/k = S_q(A)/k + S_q(B)/k + (1-q)S_q(A)S_q(B)/k^2$. This property clearly exhibits the fact that the parameter q characterizes the degree of non-extensivity of any physical system.

The generalization of quantum statistics for non-extensive systems was only accomplished, up to recent days, in an *approximate fashion*, by using two different schemes. One of them is the Asymptotic Approach (AA), of Tsallis *et al.* [5], the other one is the Factorization Approach (FA), of Büyükkılıç *et al.* [6]. The physical applications studied so far within these two approximations include the blackbody radiation [5,7,8], the Stefan-Boltzmann constant [9–11], and some aspects of the early universe physics [12,13]. Moreover, the AA has also been used in some other works such as the Bose-Einstein condensation [14], the specific heat of ^{4}He [15], thermalization of an electron-phonon system [16] and cosmology [17,18]. Although some detailed analysis on these approximate schemes [19] suggest that both schemes could be helpful in physical applications –at least, for $(1-q)$-order corrections–, this was still doubtful. A complete verification needs a comparison between the results of these approximate schemes and the exact ones. But an exact treatment of non-extensive quantum distributions was not available up to the recent papers of Rajagopal *et al.* [20,21]. Just after this work, Lenzi and Mendes have also given an exact treatment of blackbody radiation [22]. All these recent efforts enable us to make a comparison between the approximate and exact schemes, which will ultimately show whether the AA and the FA are useful or not. This will be the main purpose of this paper.

In Section 2, we review the approximate and exact results and develop an analytical method to derive

[a] e-mail: tirnakli@sci.ege.edu.tr

the explicit form of any measurable quantity within the AA. We compare the approximate and exact results in Section 3 using (i) the predictions of one of the experimental tests suggested in [20] and (ii) the blackbody radiation. Finally, we give our conclusions and final comments in Section 4.

2 Non-extensive quantum statistics

2.1 Asymptotic approach

Within the AA, namely in the $\beta(1-q) \to 1$ limit, the generalized partition function is given by [5]

$$Z_q \simeq Z_1 \left\{ 1 - \frac{1}{2}(1-q)\beta^2 \left\langle \hat{\mathcal{H}}^2 \right\rangle_1 \right\}, \tag{2}$$

from where the generalized distribution function of non-interacting bosons can be found, up to $(1-q)$-order, as

$$\begin{aligned} \langle n \rangle_q = \langle n \rangle_1 + (1-q) \langle n \rangle_1 \Bigg\{ \ln(1/Z_1) \\ + (x-\psi) \left[\frac{\langle n^2 \rangle_1}{\langle n \rangle_1} + (x-\psi) \left(\langle n^2 \rangle_1 - \frac{\langle n^3 \rangle_1}{\langle n \rangle_1} \right) \right] \Bigg\}, \end{aligned} \tag{3}$$

where $x \equiv \beta\epsilon$, $\psi \equiv \beta\mu$, and

$$\begin{aligned} \langle n \rangle_1 &= \frac{1}{\mathrm{e}^{x-\psi} - 1}, \\ \langle n^2 \rangle_1 &= \frac{\mathrm{e}^{-(x-\psi)} + \mathrm{e}^{-2(x-\psi)}}{\left[1 - \mathrm{e}^{-(x-\psi)}\right]^2}, \\ \langle n^3 \rangle_1 &= \frac{\mathrm{e}^{-(x-\psi)} + 4\mathrm{e}^{-2(x-\psi)} + \mathrm{e}^{-3(x-\psi)}}{\left[1 - \mathrm{e}^{-(x-\psi)}\right]^3} \cdot \end{aligned} \tag{4}$$

The standard $(q = 1)$ partition function is given by

$$Z_1 = \frac{1}{1 - \mathrm{e}^{-(x-\psi)}} \cdot \tag{5}$$

This approximation has found a wide range of applications up to now, however, no attempt has been made for deriving some of the thermodynamical quantities within this approach, directly using equation (3).

One aim of this paper is to provide a technique for computing, in a closed form, the kind of integrals needed to find the average number of particles within the AA. To do this, let us start by writing down the definition of the average number of particles:

$$\langle N \rangle_q = \frac{2\pi V (2mk)^{3/2} T^{3/2}}{h^3} \int_0^\infty \epsilon^{1/2} \langle n \rangle_q \, \mathrm{d}\epsilon, \tag{6}$$

where all variables have the usual meaning. Using equation (3) and the definitions of x and ψ, this expression turns out to be

$$\langle N \rangle_q = \frac{2\pi V (2mk)^{3/2} T^{3/2}}{h^3} \left[I_{\mathrm{st}} + (1-q)(I_2 + I_3) \right], \tag{7}$$

where

$$I_{\mathrm{st}} = \int_0^\infty \frac{x^{1/2}\mathrm{d}x}{\mathrm{e}^{x-\psi} - 1}, \quad I_2 = \int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{d}x}{\mathrm{e}^{x-\psi} - 1}, \tag{8}$$

and

$$\begin{aligned} I_3 = \int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{d}x}{\mathrm{e}^{x-\psi} - 1} \Bigg[\frac{\langle n^2 \rangle_1}{\langle n \rangle_1} \\ +(x-\psi) \left(\langle n^2 \rangle_1 - \frac{\langle n^3 \rangle_1}{\langle n \rangle_1} \right) \Bigg]. \end{aligned} \tag{9}$$

I_2 and I_3 are the $(q-1)$ order correction to the standard $(q = 1)$ result and here I_{st} stands for the standard integral appearing in the solution of the extensive case [23]. I_{st} and I_2 have standard forms, and could easily be solved as:

$$I_{\mathrm{st}} = \Gamma(3/2) g_{3/2}(z), \tag{10}$$

$$\begin{aligned} I_2 &= \int_0^\infty \frac{x^{5/2-1}\mathrm{d}x}{\mathrm{e}^{x-\psi} - 1} - \psi \int_0^\infty \frac{x^{3/2-1}\mathrm{d}x}{\mathrm{e}^{x-\psi} - 1} \\ &= \Gamma(5/2) g_{3/2}(z) - \psi \Gamma(3/2) g_{3/2}(z), \end{aligned} \tag{11}$$

where z is the fugacity and is defined as $z \equiv \mathrm{e}^{\beta\mu}$. On the other hand, I_3 is more involved, and it takes the form:

$$I_3 = a + b - 3c - d, \tag{12}$$

where

$$\begin{aligned} a &= \int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{d}x}{\left[\mathrm{e}^{x-\psi} - 1\right]^2}, \\ b &= \int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{e}^{x-\psi}\mathrm{d}x}{\left[\mathrm{e}^{x-\psi} - 1\right]^2}, \end{aligned} \tag{13}$$

$$\begin{aligned} c &= \int_0^\infty \frac{(x-\psi)^2 x^{1/2}\mathrm{e}^{x-\psi}\mathrm{d}x}{\left[\mathrm{e}^{x-\psi} - 1\right]^3}, \\ d &= \int_0^\infty \frac{(x-\psi)^2 x^{1/2}\mathrm{e}^{2(x-\psi)}\mathrm{d}x}{\left[\mathrm{e}^{x-\psi} - 1\right]^3} \cdot \end{aligned} \tag{14}$$

In an Appendix, we provide an analytical technique (maybe there are others) to compute each one of these integrals. Using this technique, we obtain the average number of particles as:

$$\begin{aligned} \langle N_{\mathrm{e}} \rangle_q \frac{h^3}{2\pi V (2mkT)^{3/2}} = \Gamma(3/2) g_{3/2}(z) + (q-1)\sqrt{\pi} \\ \times \left[\frac{3}{2} g_{3/2}(z) - \frac{9}{8} g_{5/2}(z) + \frac{7}{4} \psi g_{3/2}(z) \right. \\ \left. -2\psi g_{1/2}(z) - \frac{1}{2}\psi^2 g_{1/2}(z) + \frac{9}{8}\psi^2 g_{-1/2}(z) \right] \cdot \end{aligned} \tag{15}$$

Here, $\langle N_{\mathrm{e}} \rangle_q$ stands for the number of particles in the excited states $(\epsilon \neq 0)$. As in the standard case, we have separated the contribution of the state given by $\epsilon = 0$, which has zero weight in the integrals. For this level of

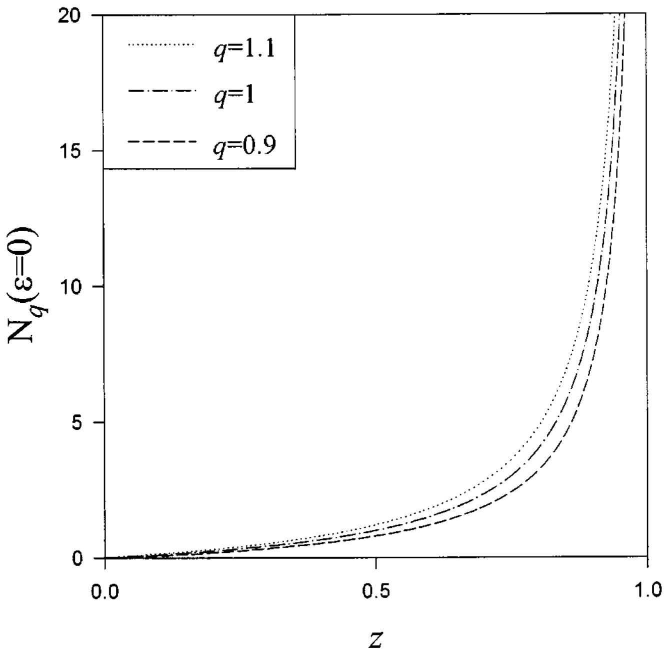

Fig. 1. $N_q(\epsilon = 0)$ of the AA as a function of z for different q values.

energy, we found,

$$\langle N\rangle_q(\epsilon=0)=\frac{z}{1-z}\left\{1+(q-1)\left[\ln z+\frac{z\ln z}{1-z}\right.\right.$$
$$\left.\left.-\frac{\ln z}{1-z}+3\frac{z(\ln z)^2}{(1-z)^2}+\frac{(\ln z)^2}{(1-z)^2}\right]\right\}. \quad (16)$$

When $z \ll 1$, all the correction terms go to zero. When $z \to 1$, some of the terms are divergent but the usual shape is unchanged. This can be seen in Figure 1.

Numerical analysis, which we show in Figure 2, illustrates that the maximum correction is attained for $z = 1$. Then, the number of particles in all excited states is bounded by,

$$\langle N_{\mathrm{e}}\rangle_q \le \frac{2\pi V(2mk)^{3/2}T^{3/2}}{h^3}\left[2.315+(q-1)4.27\right]. \quad (17)$$

It is worth noticing that the AA is such that not all terms in the $(1-q)$ correction are positive (or negative, depending on the choice of q) definite. Moreover, their maximum values are not always attained at middle points of the interval of interest, and although they do have bounded expressions, the maximum correction is obtained only for $z = 1$. This differs from what happened in the FA, where each term had a maximum value within the interval of interest [24]. The order of magnitude of the maximum correction is, however, the same in both approximations.

Any interested reader could easily apply the same technique, which we introduced in the Appendix, to compute any other thermodynamical quantity, whenever it is needed.

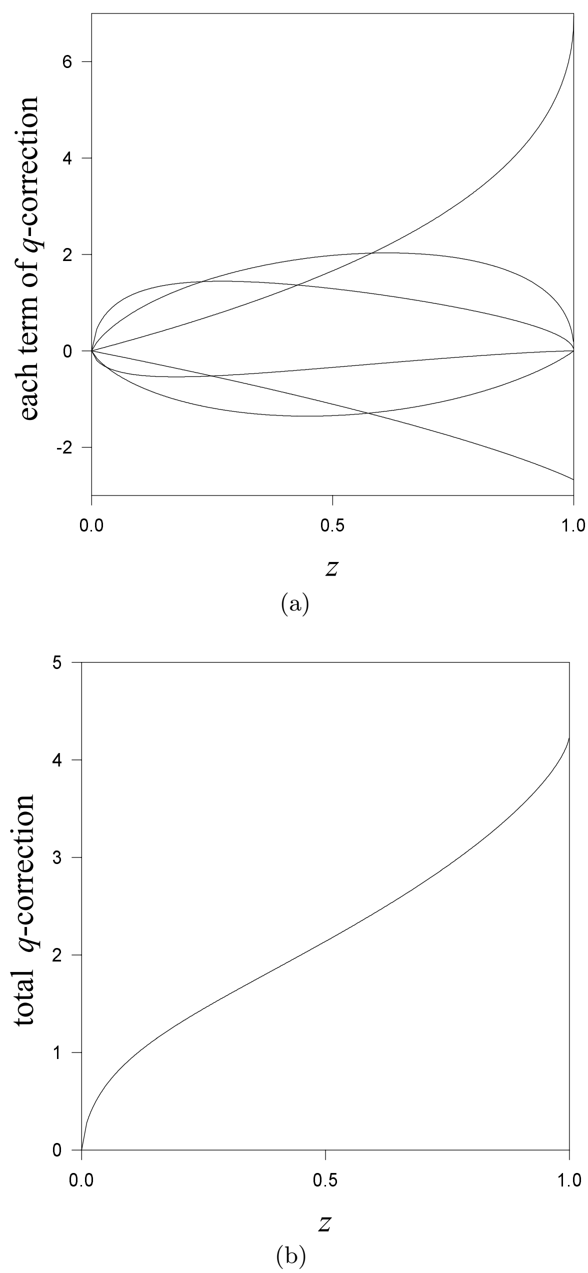

Fig. 2. (a) The contribution of the different terms that enter the $(q-1)$ correction to the average number of bosons within the AA. On the right corner of the figure, the curves corresponds to the following order: first, fourth, sixth, fifth, second and third term. (b) The total $(q-1)$ correction to the average number of bosons within the AA. Its maximum possible value is attained at $z = 1$, and the correction goes as $\pi^{1/2}(3/2\zeta(3/2) - 9/8\zeta(5/2)) = 4.27$.

2.2 Factorization approach

Within the FA [6], the generalized distribution function of bosons is given, at $(1-q)$ order, by [24]

$$\langle n\rangle_q=\langle n\rangle_1+(q-1)\frac{(x-\psi)^2\mathrm{e}^{x-\psi}}{2\left(\mathrm{e}^{x-\psi}-1\right)^2}, \quad (18)$$

where $\langle n \rangle_1$, x and ψ have the same definitions as before. At this point, the remarkably simpler form of this result, when compared to the result of the AA (Eq. (3)), is worth emphasizing.

In this context, we have found general expressions for some thermodynamical quantities of bosons and fermions [24]; here we quote only the average number of particles for bosons, since it will be adequate for our proposed comparison[1]:

$$\langle N_{\mathrm{e}} \rangle_q \leq \frac{2\pi V (2mk)^{3/2} T^{3/2}}{h^3} \left[2.315 + (q-1)3.079\right]. \quad (19)$$

2.3 The exact result

Although the results of the AA and the FA have been successfully used in a wide range of physical applications, an exact treatment of non-extensive quantum statistics was lacking until the recent work of Rajagopal, Mendes and Lenzi [20,21]. In their analysis, they have given the many-particle q-Green function in terms of a parametric contour integral over a kernel, multiplied by the usual grand canonical one particle Green function which now depends on q. They managed to obtain exact expressions for thermodynamical quantities, such as $\langle N \rangle_q$.

To proceed further, let us quote here some of the results of [20,21]. Rajagopal *et al.* have used the general contour integral of the form,

$$b^{1-z} \frac{\mathrm{i}}{2\pi} \int_C \mathrm{d}u \exp(-bu)(-u)^{-z} = \frac{1}{\Gamma(z)}, \quad (20)$$

with $b > 0$ and Re $z > 0$, and where the contour C starts from $+\infty$ on the real axis, encircles the origin once counterclockwise and returns to $+\infty$. Using the q-Green functions, and after some cumbersome algebra, they finally obtain (for bosons)

$$\langle N \rangle_q = V \int_C \mathrm{d}u K_q^{(2)}(u) \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \times \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{Z_1(-\beta(1-q)u, \mu)}{\left[\mathrm{e}^{-\beta(1-q)u(\omega-\mu)} - 1\right]} A(\mathbf{p};\omega), \quad (21)$$

where D is the dimension of space, $A(\mathbf{p};\omega)$ is the spectral weight function and

$$K_q^{(2)}(u) = \mathrm{i} \frac{\Gamma(1/(1-q))}{2\pi (Z_q)^q} \exp(-u)(-u)^{-1/(1-q)}, \quad (22)$$

and

$$Z_q(\beta,\mu) = \int_C \mathrm{d}u K_q^{(1)}(u) Z_1(-\beta u(1-q), \mu). \quad (23)$$

[1] We take advantage here to signal out a mistake in the last equation of reference [24], where the correction appears to be proportional to 0.886 $(q-1)$ and should have a minus sign in front of it [25].

This exact expression for the average number of particles finally gives us the opportunity to make a comparison between the exact and the approximate results.

3 Exact and approximate results

3.1 Bose-Einstein condensate

One of possible experimental tests of the validity of the q-framework is based on a recent work on Bose-Einstein condensation of a small number of atoms (of the order of 100 to 170), confined to a small region of space by magnetic trapping [26]. By taking free particle spectral weight function, namely $A(\mathbf{p};\omega) = 2\pi\delta(\omega - \mathbf{p}^2/2m)$, near the Bose-Einstein condensation, they have found

$$\frac{\langle N \rangle_q}{\langle N \rangle_1} \simeq \left[\frac{(T_{\mathrm{c}})_q}{(T_{\mathrm{c}})_1}\right]^{3/2} \frac{\Gamma\left(\frac{2-q}{1-q}\right)}{(1-q)^{1/2}\Gamma\left(\frac{2-q}{1-q}+\frac{1}{2}\right)} \times \left\{1 + \frac{\langle N \rangle_1}{(1-q)^{3/2}} \frac{\zeta(5/2)}{\zeta(3/2)} \left[\frac{(T_{\mathrm{c}})_q}{(T_{\mathrm{c}})_1}\right]^{3/2} \times \left[\frac{\Gamma\left(\frac{2-q}{1-q}+\frac{1}{2}\right)}{\Gamma\left(\frac{2-q}{1-q}+2\right)} - q\frac{\Gamma\left(\frac{2-q}{1-q}\right)}{\Gamma\left(\frac{2-q}{1-q}+\frac{3}{2}\right)}\right]\right\}. \quad (24)$$

Here, the $\simeq$ sign reflects that this equation is valid near the Bose-Einstein condensation, which does not change the fact that it is an exact result. We may now expand this expression in powers of $(q-1)$. Up to first order,

$$\frac{\langle N \rangle_q}{\langle N \rangle_1} \simeq \left[\frac{(T_{\mathrm{c}})_q}{(T_{\mathrm{c}})_1}\right]^{3/2} \left\{1 + (q-1)\left(0.456 - 0.023 \langle N \rangle_1 \left[\frac{(T_{\mathrm{c}})_q}{(T_{\mathrm{c}})_1}\right]^{3/2}\right)\right\}. \quad (25)$$

The two previous equations deviate from each other very soon when $(q-1)^2$ is not negligible.

Let us now derive similar expressions for the AA and FA in order to compare them with equations (24, 25). The condition for the appearance of Bose-Einstein condensation can be expressed as

$$\langle N \rangle_q > \langle N_{\mathrm{e}} \rangle_q. \quad (26)$$

Alternatively, with constant $\langle N \rangle_q$ and V, using equation (17) for the AA and equation (19) for the FA, this condition can be recast in the form

$$T < (T_{\mathrm{c}})_q = \frac{h^2}{(2\pi)^{3/2} 2mk} \left\{\frac{\langle N \rangle_q}{V\left[2.315 + (q-1)\kappa\right]}\right\}^{2/3}, \quad (27)$$

where $\kappa = 4.27$ for the AA and $\kappa = 3.078$ for the FA. Then, we can organize these expressions to give

$$\frac{\langle N\rangle_q}{\langle N\rangle_1} = \left[\frac{(T_{\rm c})_q}{(T_{\rm c})_1}\right]^{3/2} \frac{[2.315 + (q-1)\kappa]}{2.315}. \tag{28}$$

Equations (17, 19) have corrections which are trivial (not depending on z) just because we have approximated them: the actual complete results are equations (15, 16) for the AA, while those for the FA can be found in our previous paper [24]. We managed the dependence on z in order to obtain an upper bound for the corrections and simplify the analysis that follows. Differences between equations (28, 25) are worth noticing: the later depends on $(T_{\rm c})_q$ and $\langle N\rangle_1$ in a much stronger way. However, as we shall see, for q close to 1 these differences are not important.

We would now like to choose physically suitable q values. An early Universe test based on the FA has shown [13] to produce a bound $|q-1| \leq 4.01 \times 10^{-3}$, thus we have $q = 0.996$. The other q value which we use comes from a very recent work on pion transverse-momentum correlations in Pb-Pb high-energy nuclear collisions [27]. In that work, a deviation of $|q-1| = 0.015$ from the standard statistics is found to be sufficient for eliminating the puzzling discrepancy between theoretical calculations and experimental data [27]. Thus, we shall use $q = 0.985$ (in fact, in [27], $q = 1.015$ has been used, but since the exact result is given for $q < 1$ values, we must take $q = 0.985$, which has the same $|q-1|$ deviation). In Figure 3 we plot $\langle N\rangle_q / \langle N\rangle_1$ *versus* $(T_{\rm c})_q / (T_{\rm c})_1$ for two representative values of $\langle N\rangle_1$, and the two quoted values of q. However, note again that in our approximated schemes, $\langle N\rangle_q / \langle N\rangle_1$ as a function of $(T_{\rm c})_q / (T_{\rm c})_1$ is in fact independent of the particular value of $\langle N\rangle_1$. From Figure 3, the following conclusions can be drawn: At the order of such q values, the AA and the FA are almost the same, and in $(1-q)$-order correction, any of them could be used with the same confidence (maybe the FA would be preferable due to its remarkably simpler form). Only in those situations of extremely high experimental precision one could distinguish between the exact and approximate results.

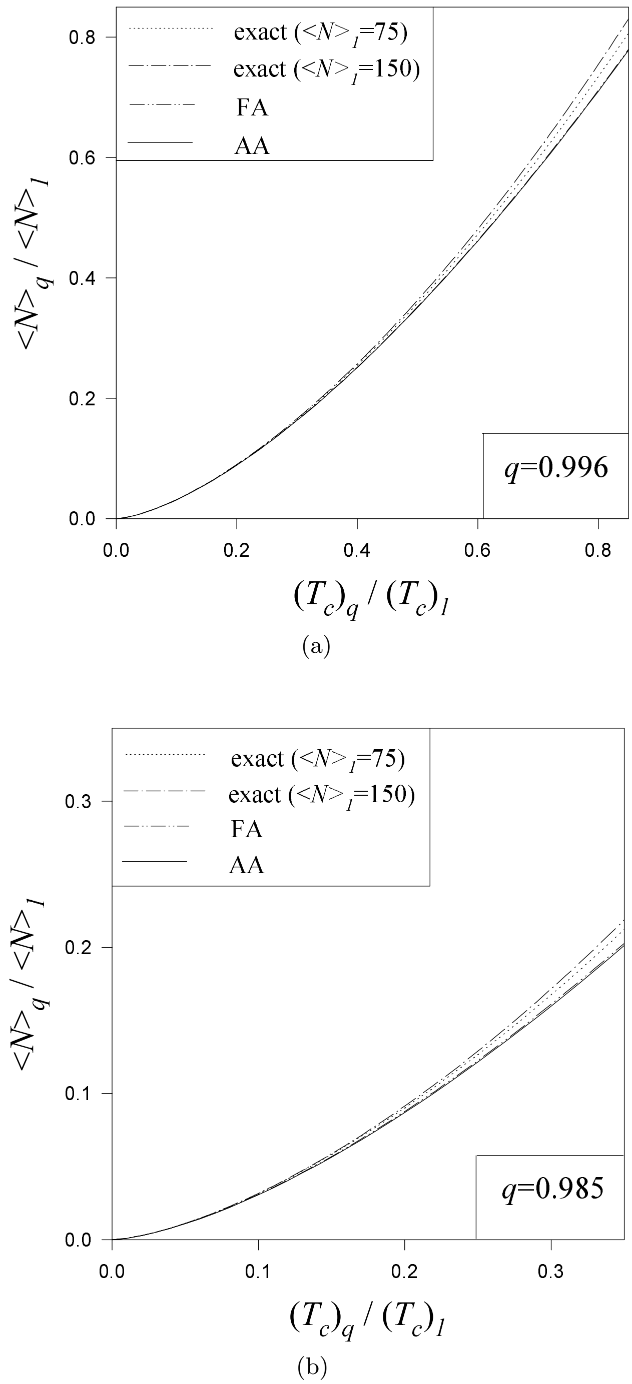

Fig. 3. Bose-Einstein condensation: plot of $\langle N\rangle_q / \langle N\rangle_1$ as a function of $(T_{\rm c})_q / (T_{\rm c})_1$ for (a) $q = 0.996$ and (b) $q = 0.985$.

3.2 Blackbody radiation

Very recently, an exact analysis of the blackbody radiation within the q-framework has been given [22]. This exact analysis gives the generalization of the Stefan-Boltzmann law as

$$U_q = \frac{3kT\xi_3}{Z_q^q} \sum_{m=0}^{\infty} \frac{\xi_3^m}{m!} \frac{\Gamma[(2-q)/(1-q)]}{\Gamma[(2-q)/(1-q)+3(m+1)]}, \tag{29}$$

where

$$Z_q = \sum_{m=0}^{\infty} \frac{\xi_3^m}{m!} \frac{\Gamma[(2-q)/(1-q)]}{\Gamma[(2-q)/(1-q)+3m]}, \tag{30}$$

and

$$\xi_3 = \frac{4\Gamma(3)\zeta(4)}{[2\pi^{1/2}(1-q)]^3 \Gamma(3/2)} \left(\frac{2\pi V^{1/3} kT}{hc}\right)^3. \tag{31}$$

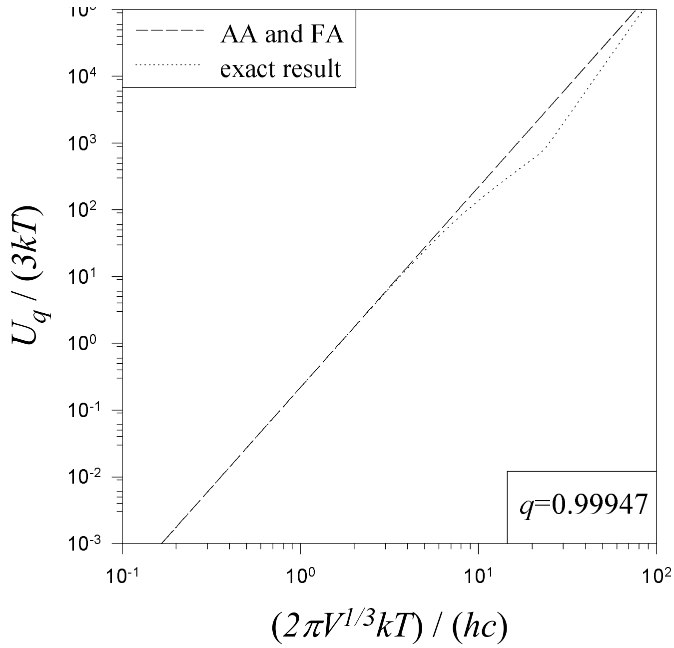

Fig. 4. Blackbody radiation: Internal energy *versus* $(2\pi kT)/(hc)$ for $q = 0.99947$.

Let us now recall the Stefan-Boltzmann law derived by using the AA [9,10] and the FA [11]:

$$U_q = \frac{8\pi k^4 T^4 V}{c^3 h^3}\left[6.4939 - (1-q)\theta\right] \tag{32}$$

where $\theta = 40.018$ for the AA and $\theta = 62.215$ for the FA.

For the comparison of the exact and approximate Stefan-Boltzmann laws, we again try to choose a value of q which is in accordance with the blackbody radiation. The possible q-correction could be at the order of 10^{-4} or 10^{-5}. Thus, here we shall use again the largest deviation predicted for the q-correction [9], namely $|q-1| \le 5.3\times 10^{-4}$, which gives $q = 0.99947$. In Figure 4 we present the behaviour of the exact (Eq. (29)) and the approximate results (Eq. (32)) for $q = 0.99947$. It is seen from the figure that for such order of q-correction the approximate results are very close to the standard ($q = 1$) case without exhibiting any curvature, contrary to the exact result.

4 Final remarks

We have managed to develop an analytical technique to express thermodynamical quantities for the asymptotic approach of quantum distribution functions. We have shown that, for simple boson systems, and for all q-values admitted by the existing bounds, both approximate schemes (the AA and the FA) are in agreement with the exact result (see figures). The magnitude of the deviation is quantified in previous formulae and could be seen if there is enough experimental precision. Otherwise, the simpler form that the factorization approach exhibits makes a case for its use as a standard and safe procedure for $(1-q)$-order corrections.

U.T. acknowledges the partial support of BAYG-C program of TUBITAK (Turkish Agency) and CNPq (Brazilian agency) as well as the support from Ege University Research Fund under the project number 98FEN025. D.F.T. acknowledges support from CONICET and Fundación Antorchas and wishes to thank A. Lavagno for sending him his valuable Ph.D. thesis. We also thank A.K. Rajagopal, A. Wang, A. Erzan and an anonymous referee for useful comments.

Appendix A

The calculation of the integral a can be done as follows. Let us define, introducing an extra parameter m,

$$I = \int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{d}x}{\mathrm{e}^{x-\psi} - m}\cdot \tag{A.1}$$

Then, we could write,

$$a = \left[\frac{\mathrm{d}I}{\mathrm{d}m}\right]_{m=1} = \left[\int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{d}x}{\left(\mathrm{e}^{x-\psi} - m\right)^2}\right]_{m=1}. \tag{A.2}$$

Defining $m^{-1} \equiv \mathrm{e}^{\phi}$, it is easy to write,

$$\begin{aligned} a_m &= \frac{\mathrm{d}}{\mathrm{d}m}\int_0^\infty \frac{(x-\psi)x^{1/2}\mathrm{d}x}{m\left(m^{-1}\mathrm{e}^{x-\psi} - 1\right)} \\ &= \frac{\mathrm{d}}{\mathrm{d}m}\left\{\frac{1}{m}\left[\int_0^\infty \frac{x^{3/2}\mathrm{d}x}{\mathrm{e}^{x-\psi'} - 1} - \psi\int_0^\infty \frac{x^{1/2}\mathrm{d}x}{\mathrm{e}^{x-\psi'} - 1}\right]\right\}, \end{aligned} \tag{A.3}$$

where $\psi' \equiv \psi - \phi$. This let us to obtain,

$$a_m = \frac{\mathrm{d}}{\mathrm{d}m}\left[\frac{1}{m}\left(\Gamma(5/2)g_{5/2}(\psi') - \psi\Gamma(3/2)g_{3/2}(\psi')\right)\right]. \tag{A.4}$$

Finally this gives us the solution of the integral a:

$$\begin{aligned} a = \Gamma(5/2)g_{3/2}(z) - \Gamma(5/2)g_{5/2}(z) \\ - \psi\Gamma(3/2)g_{1/2}(z) + \psi\Gamma(3/2)g_{3/2}(z). \end{aligned} \tag{A.5}$$

For the calculation of c, we may conveniently define a_l as,

$$a_l = \int \frac{(x-\psi)x^{1/2}\mathrm{d}x}{\left[\mathrm{e}^{l(x-\psi)} - 1\right]^2} \tag{A.6}$$

and derive with respect to l to obtain

$$\left[\frac{\mathrm{d}a_l}{\mathrm{d}l}\right]_{l=1} = -2\int \frac{(x-\psi)^2 x^{1/2}\mathrm{e}^{x-\psi}\mathrm{d}x}{(\mathrm{e}^{x-\psi}-1)^3} = -2c. \quad \text{(A.7)}$$

Now, to compute a_l, we may change variables as follows:

$$\tilde{x} = lx, \quad \tilde{\psi} = l\psi. \quad \text{(A.8)}$$

We then obtain,

$$a_l = \frac{1}{l^{5/2}}\int \frac{(\tilde{x}-\tilde{\psi})\tilde{x}^{1/2}\mathrm{d}\tilde{x}}{(\mathrm{e}^{(\tilde{x}-\tilde{\psi})}-1)^2}$$
$$= \frac{1}{l^{5/2}}\left\{\Gamma(5/2)\left[g_{3/2}(\tilde{\psi}) - g_{5/2}(\tilde{\psi})\right] + \tilde{\psi}\Gamma(3/2)\left[g_{3/2}(\tilde{\psi}) - g_{1/2}(\tilde{\psi})\right]\right\}. \quad \text{(A.9)}$$

Deriving with respect to l, we get

$$\frac{\mathrm{d}a_l}{\mathrm{d}l} = -\frac{5}{2l^{7/2}}\left\{\Gamma(5/2)\left[g_{3/2}(\tilde{\psi}) - g_{5/2}(\tilde{\psi})\right] + \tilde{\psi}\Gamma(3/2)\left[g_{3/2}(\tilde{\psi}) - g_{1/2}(\tilde{\psi})\right]\right\}$$
$$+ \frac{1}{l^{5/2}}\left\{\Gamma(5/2)\left[\psi g_{1/2}(\tilde{\psi}) - \psi g_{3/2}(\tilde{\psi})\right] + \psi\Gamma(3/2)\left[g_{3/2}(\tilde{\psi}) - g_{1/2}(\tilde{\psi})\right] + \tilde{\psi}\Gamma(3/2)\left[\psi g_{1/2}(\tilde{\psi}) - \psi g_{-1/2}(\tilde{\psi})\right]\right\}. \quad \text{(A.10)}$$

From this equation, the integral c can be obtained by making $l=1$ and $\tilde{\psi}=\psi$. Here, we should note that, in all previous calculations, we have used (i) the result

$$g_{n-1}(z) = z\frac{\partial}{\partial z}\left[g_n(z)\right] = \frac{\partial}{\partial \ln(z)}\left[g_n(z)\right], \quad \text{(A.11)}$$

and (ii) Robinson's power series representation [28] (which is valid for *all n values*):

$$g_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{l=0}^{\infty}\frac{(-)^l}{l!}\zeta(n-l)\alpha^l, \quad \text{(A.12)}$$

for the g_n functions, where $\alpha = -\ln(z)$ and ζ is the Riemann zeta function. Using this result, one can recover the relationship for the derivatives of g_n functions, and use it to evaluate $g_{-1/2}$. We finally obtain the result for the integral c:

$$c = \frac{5}{4}\Gamma(5/2)g_{3/2}(\psi) - \frac{5}{4}\Gamma(5/2)g_{5/2}(\psi) + \frac{5}{4}\Gamma(3/2)\psi g_{3/2}(\psi) - \frac{5}{4}\Gamma(3/2)\psi g_{1/2}(\psi)$$
$$- \frac{1}{2}\Gamma(5/2)\psi g_{1/2}(\psi) + \frac{1}{2}\Gamma(5/2)\psi g_{3/2}(\psi) - \frac{1}{2}\Gamma(3/2)\psi g_{3/2}(\psi) + \frac{1}{2}\Gamma(3/2)\psi g_{1/2}(\psi)$$
$$- \frac{1}{2}\Gamma(3/2)\psi^2 g_{1/2}(\psi) + \frac{1}{2}\Gamma(3/2)\psi^2 g_{-1/2}(\psi). \quad \text{(A.13)}$$

We proceed further to compute integral b. To do so, we apply the following procedure. Let us define a new integral with an extra parameter, such that

$$a_m = \int_0^{\infty}\frac{x^{1/2}\mathrm{d}x}{\mathrm{e}^{m(x-\psi)}-1} = \frac{1}{m^{3/2}}\int_0^{\infty}\frac{\tilde{x}^{1/2}\mathrm{d}\tilde{x}}{\mathrm{e}^{(\tilde{x}-\tilde{\psi})}-1}$$
$$= \frac{1}{m^{3/2}}\Gamma(3/2)g_{3/2}(\tilde{\psi}), \quad \text{(A.14)}$$

where we have used the change of variable $\tilde{x} = mx$ and $\tilde{\psi} = m\psi$. If we now derive with respect to m, we obtain,

$$\left[\frac{\mathrm{d}a_m}{\mathrm{d}m}\right]_{m=1} = \left[-\int_0^{\infty}\frac{(x-\psi)x^{1/2}\mathrm{e}^{m(x-\psi)}\mathrm{d}x}{(\mathrm{e}^{m(x-\psi)}-1)^2}\right]_{m=1} = -b. \quad \text{(A.15)}$$

Since we have,

$$\frac{\mathrm{d}a_m}{\mathrm{d}m} = -\frac{3}{2m^{5/2}}\Gamma(3/2)g_{3/2}(\tilde{\psi}) + \frac{1}{m^{3/2}}\Gamma(3/2)\psi g_{1/2}(\tilde{\psi}), \quad \text{(A.16)}$$

it is easy to write down the solution of the integral b:

$$b = \frac{3}{2}\Gamma(3/2)g_{3/2}(\psi) - \Gamma(3/2)\psi g_{1/2}(\psi). \quad \text{(A.17)}$$

Once the integral b is calculated, the integral d can be obtained as follows. Let us define a_m as above. Deriving it twice with respect to the parameter m, we obtain

$$\left[\frac{\mathrm{d}^2 a_m}{\mathrm{d}m^2}\right] = 2d - I_{\mathrm{new}}, \quad \text{(A.18)}$$

where,

$$I_{\mathrm{new}} = \int_0^{\infty}\frac{(x-\psi)^2 x^{1/2}\mathrm{e}^{m(x-\psi)}\mathrm{d}x}{\left[\mathrm{e}^{m(x-\psi)}-1\right]^2}. \quad \text{(A.19)}$$

It is easy to compute this integral with a similar trick. We need to define, with usual notation,

$$I_{\mathrm{ext}} = \int_0^{\infty}\frac{(x-\psi)x^{1/2}\mathrm{d}x}{\left[\mathrm{e}^{l(x-\psi)}-1\right]}$$
$$= \frac{1}{l^{5/2}}\left[\Gamma(5/2)g_{5/2}(\tilde{\psi}) - \tilde{\psi}\Gamma(3/2)g_{3/2}(\tilde{\psi})\right] \quad \text{(A.20)}$$

and derive it with respect to l. Further evaluation in $l=1$ reproduces I_{new}:

$$I_{\mathrm{new}} = \frac{5}{2}\Gamma(5/2)g_{5/2}(\psi) - \psi\frac{3}{2}\Gamma(3/2)g_{3/2}(\psi) - \psi\Gamma(5/2)g_{3/2}(\psi) + \psi^2\Gamma(3/2)g_{1/2}(\psi). \quad \text{(A.21)}$$

Thus, we finally have the solution of the integral d:

$$d = \frac{15}{8}\Gamma(3/2)g_{3/2}(\psi) - \frac{3}{2}\psi\Gamma(3/2)g_{1/2}(\psi) + \frac{3}{4}\psi^2\Gamma(3/2)g_{-1/2}(\psi) + \frac{5}{4}\Gamma(5/2)g_{5/2}(\psi)$$
$$- \psi\frac{3}{4}\Gamma(3/2)g_{3/2}(\psi) - \frac{1}{2}\psi\Gamma(5/2)g_{3/2}(\psi) + \frac{1}{2}\psi\Gamma(3/2)g_{3/2}(\psi) + \frac{1}{2}\psi^2\Gamma(3/2)g_{1/2}(\psi). \quad \text{(A.22)}$$

References

1. C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
2. E.M.F. Curado, C. Tsallis, J. Phys. A **24**, L69 (1991); corrigenda **24**, 3187 (1991); **25**, 1019 (1992).
3. C. Tsallis, Physica A **221**, 277 (1995); C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A **261**, 534 (1998).
4. `http://tsallis.cat.cbpf.br/biblio.htm`
5. C. Tsallis, F.C. Sa Barreto, E.D. Loh, Phys. Rev. E **52**, 1447 (1995).
6. F. Büyükkılıç, D. Demirhan, Phys. Lett. A **181**, 24 (1993); F. Büyükkılıç, D. Demirhan, A. Gulec, Phys. Lett. A **197**, 209 (1995).
7. A.B. Pinheiro, I. Roditi, Phys. Lett. A **242**, 296 (1998).
8. U. Tırnaklı, F. Büyükkılıç, D. Demirhan, Physica A **240**, 657 (1997).
9. A.R. Plastino, A. Plastino, H. Vucetich, Phys. Lett. A **207**, 42 (1995).
10. Q.A. Wang, A. Le Méhauté, Phys. Lett. A **237**, 28 (1997); Q.A. Wang, L. Nivanen, A. Le Méhauté, Physica A **260**, 490 (1998).
11. U. Tırnaklı, F. Büyükkılıç, D. Demirhan, Phys. Lett. A **245**, 62 (1998).
12. D.F. Torres, H. Vucetich, A. Plastino, Phys. Rev. Lett. **79**, 1588 (1997); Erratum **80**, 3889 (1998).
13. U. Tırnaklı, D.F. Torres, Physica A **268**, 225 (1999).
14. S. Curilef, Phys. Lett. A **218**, 11 (1996).
15. S. Curilef, A.R.R. Papa, Int. J. Mod. Phys. B **11**, 2303 (1997).
16. I. Koponen, Phys. Rev. E **55**, 7759 (1997).
17. D.F. Torres, Physica A **261**, 512 (1998).
18. D.F. Torres, H. Vucetich, Physica A **259**, 397 (1998).
19. Q.A. Wang, A. Le Méhauté, Phys. Lett. A **235**, 222 (1997).
20. A.K. Rajagopal, R.S. Mendes, E.K. Lenzi, Phys. Rev. Lett. **80**, 3907 (1998).
21. E.K. Lenzi, R.S. Mendes, A.K. Rajagopal, Phys. Rev. E **59**, 1398 (1999).
22. E.K. Lenzi, R.S. Mendes, Phys. Lett. A **250**, 270 (1998).
23. R.K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1985).
24. D.F. Torres, U. Tırnaklı, Physica A **261**, 499 (1998).
25. Q.A. Wang (private communications, 1999).
26. M.H. Anderson, J.R. Ensher, M.R. Mathews, C.E. Wieman, E.A. Cornell, Science **269**, 198 (1995).
27. G. Kaniadakis, A. Lavagno, M. Lissia, P. Quarati, *Non-extensive statistical effects in nuclear physics problems*, Perspectives on Theoretical Nuclear Physics, edited by A. Fabrocini, G. Pisent, S. Rosati (1999), p.293.
28. J.E. Robinson, Phys. Rev. **83**, 679 (1951).